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The Distribution Functions of Finitely Additive Vector Measures over $\mathbb{R}^q$, I

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1. INTRODUCTION

Let $\mathscr{X}$ be a Banach space and $\mathscr{P}^q$ be the pre-ring of bounded half-open half-closed sub-intervals of $\mathbb{R}^q$, $q \geqslant 1$. Our concern is with finitely additive (f.a.) measures $\rho(\cdot)$ on $\mathscr{P}^q$ to $\mathscr{X}$. Observing that even when $\mathscr{X} = \mathbb{R}$ the distribution function $x(\cdot)$ of such a measure $\rho(\cdot)$ can be non-measurable (cf. 2.6), our goal is to show that this function $x(\cdot)$ will be continuous on $\mathbb{R}^q$, if and only if $\rho(\cdot)$ is ultra-regular in a sense to be defined, cf. Def. 2.2(b) and Thm. 2.7.

For non-negative real-valued f.a. measures $\mu(\cdot)$ on $\mathscr{P}^q$, the concepts of regularity and distribution function are well known. Regularity is defined so as to ensure that a regular f.a. non-negative real-valued measure μ on any pre-ring is countably additive (c.a.) thereon. This basic proposition and the relevant notion of regularity go back to von Neumann in the early 1930s [3, p. 98, No. 120]. It is easily seen that the imposition of regularity on such a measure μ on the pre-ring $\mathscr{P}^q$ makes its distribution function $x(\cdot)$ right-continuous on $\mathbb{R}^q$. It does not make it continuous, for the Hahn–Caratheodory extension $\tilde{\mu}$ of such a regular (and therefore c.a.) μ on $\mathscr{P}^q$ can possess atoms, and $x(\cdot)$ will be discontinuous at $t \in \mathbb{R}^q$ if and only if $\{t\}$ is an atom of $\tilde{\mu}$. Note that the pre-ring $\mathscr{P}^q$ is atomless ($\{t\} \notin \mathscr{P}^q$); therefore the condition that μ is atomless on $\mathscr{P}^q$ does not ensure the continuity of f on $\mathbb{R}^q$. (Just consider the μ on $\mathscr{P}$ given by the right-continuous increasing function f: $f(t) = t$ for $t < 0$, and $f(t) = t + 1$ for $t \geqslant 0$. This μ on $\mathscr{P}$ is atomless.)

For $\mathscr{X}$-valued measures the corresponding issues are more complex. In the first place, the concept of the distribution function $x(\cdot)$ on $\mathbb{R}^q$ of a f.a. $\mathscr{X}$-valued measure $\rho(\cdot)$ on $\mathscr{P}^q$ is not available in the literature, and has to be defined. Secondly, the classical notion of regularity has to be reformulated in a vectorial setting, as has been done by Dinculeanu [1, pp. 302–303], cf. Def. 2.2(a) below. However, the generalization of the von Neumann proposition, to wit:

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$$\rho(\cdot) \text{ is f.a. \& regular}^1 \text{ on } \mathscr{P}^q \Rightarrow \rho(\cdot) \text{ is c.a. on } \mathscr{P}^q,$$

fails even for $\mathscr{X} = \mathbb{R}$. Thus a regular f.a. $\mathscr{X}$-valued measure on $\mathscr{P}^q$ need not be c.a. on $\mathscr{P}^q$ and may have no Hahn–Caratheodory extension to the σ-algebra of Borel subsets of $\mathbb{R}^q$. It is now a new and non-obvious result that $\rho(\cdot)$ is regular, if and only if its distribution function $x(\cdot)$ is right-continuous on $\mathbb{R}^q$. Also non-obvious is how the regularity requirement has to be further strengthened in order to make $x(\cdot)$ continuous on $\mathbb{R}^q$. It is in the continuity of $x(\cdot)$ (not just its right-continuity) that we are interested. For we want to know which $\mathscr{X}$-valued *stationary* measures correspond to (continuous) helical varieties in $\mathscr{X}$, cf. Masani [2].

2. Regularity, Ultra-Regularity and Distribution Function

We shall adhere to the following notation:

$$(2.1)\ \begin{cases} \text{(a)} & \mathscr{X} \text{ is a Banach space over the field } \mathbb{F};^2 \\ \text{(b)} & \Omega \text{ be a Hausdorff space;} \\ \text{(c)} & \mathscr{P} \text{ is a pre-ring of subsets of } \Omega; \\ \text{(d)} & \mathrm{FA}(\mathscr{P}, \mathscr{X}) \text{ is the set of all additive measures on } \mathscr{P} \text{ with values in } \mathscr{X}. \end{cases}$$

The concepts of regular and ultra-regular measure, the first of which is due to Dinculeanu [1, pp. 302–303], are defined as follows:

2.2. Def. Let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}, \mathscr{X})$. Then we say that

(a) $\rho(\cdot)$ is *regular* on $\mathscr{P}$, iff $\forall P \in \mathscr{P}$ and $\forall \varepsilon > 0$, $\exists$ a compact set K of Ω and $\exists$ an open set U of Ω such that

$$K \subseteq P \subseteq U$$

and

$$Q \in \mathscr{P} \;\&\; K \subseteq Q \subseteq U \Rightarrow |\rho(Q) - \rho(P)| < \varepsilon.$$

(b) $\rho(\cdot)$ is *ultra-regular* on $\mathscr{P}$, iff $\forall P \in \mathscr{P}$ and $\forall \varepsilon > 0$, $\exists$ a compact set K of Ω and $\exists$ an open set U of Ω such that[3]

$$K \subseteq \operatorname{int} P \subseteq \operatorname{cls} P \subseteq U$$

[1] In the sense of Def. 2.2(a).

[2] In this paper $\mathbb{F}$ will refer to either the real number field $\mathbb{R}$ or to the complex number field $\mathbb{C}$, and $\mathbb{Z}$ to the set of all integers $\mathbb{Z}_+$, $\mathbb{R}_+$ and $\mathbb{Z}_{0+}$, $\mathbb{R}_{0+}$ will denote the subsets of positive elements and the subsets of non-negative elements of $\mathbb{Z}$ and $\mathbb{R}$.

[3] "int" and "cls" stand for the interior and closure.

and

$$Q \in \mathscr{P} \;\&\; K \subseteq Q \subseteq U \Rightarrow |\rho(Q) - \rho(P)| < \varepsilon.$$

Now let $\rho \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$, where $\mathscr{P}^q$ is the pre-ring of bounded half-open, half-closed subintervals of $\mathbb{R}^q$. The conventional definition of distribution function $x(\cdot)$ of $\rho(\cdot)$, which for $q = 1$ reads

$$x(t) = \rho(-\infty, t] = \lim_{a \to -\infty} \rho(a, t], \qquad t \in \mathbb{R},$$

is not available, since the RHS is meaningless without further assumptions on $\rho(\cdot)$. We must instead define $x(\cdot)$ by the equations

$$x(t) = -\rho(t, c], \quad t < c, \;\&\; x(t) = \rho(c, t], \quad t \geqslant c. \tag{1}$$

where $c \in \mathbb{R}$ is a fixed "center," chosen in advance. A convenient choice for c is 0. Our first task is to extend this definition to measures $\rho(\cdot)$ over $\mathbb{R}^q$ for $q > 1$. For this it is convenient to first extend to $\mathbb{R}^q$ certain notation that is familiar for $\mathbb{R}$:

2.3. *Notation.* (a) $\forall m, n \in \mathbb{Z}$ such that $m < n$,

$$[m, n] \underset{d}{=} \{m, m+1, ..., n\}, \qquad [m, n) \underset{d}{=} \{m, m+1, ..., n-1\};$$

$(m, n]$ and (m, n) are defined similarly.

(b) For $a = (a_1, ..., a_q)$ and $b = (b_1, ..., b_q) \in \mathbb{R}^q$ we shall write $a < b$ iff $\forall i \in [1, q]$, $a_i < b_i$, and we similarly define $a \leqslant b$.

(c) $\forall t = (t_1, ..., t_q) \in \mathbb{R}^q$ we shall write

$$t^- \underset{d}{=} (t_1^-, ..., t_q^-), \qquad t^+ \underset{d}{=} (t_1^+, ..., t_q^+), \qquad \operatorname{sgn} t \underset{d}{=} \prod_{i=1}^{q} \operatorname{sgn} t_i,$$

where for $\lambda \in \mathbb{R}$, $\lambda^- = -\min\{\lambda, 0\}$, $\lambda^+ = \max\{\lambda, 0\}$, and $\operatorname{sgn} \lambda$ is $+1$, 0 or -1 according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ (sgn stands for "signum.")

(d) For $a, b \in \mathbb{R}^q$ with $a \leqslant b$, we let

$$(a, b] \underset{d}{=} \{t : t \in \mathbb{R}^q \;\&\; a \leqslant t \leqslant b\},$$

and similarly define $[a, b]$, $[a, b)$, (a, b).

(e) We let

$$\mathscr{P} \underset{d}{=} \{(a, b] : a, b] : a, b \in \mathbb{R} \;\&\; a \leqslant b\},$$
$$\mathscr{P}^q \underset{d}{=} \{(a, b] : a, b \in \mathbb{R} \;\&\; a \leqslant b\}.$$

Our notation $\mathscr{P}^q$ is suggested by the fact that

$$a, b \in \mathbb{R}^q \ \&\ a < b \Rightarrow (a, b] = (a_1, b_1] \times \cdots \times (a_q, b_q]. \tag{2.4}$$

Obviously, $\forall t \in \mathbb{R}^q$, $-t^- \leqslant t \leqslant t^+$. For $q = 1$ it is clear that for all $t \in \mathbb{R}$, the interval $(-t^-, t^+]$ is $(0, t]$ or $(t, 0]$ according as $t \geqslant 0$ or $t \leqslant 0$, and so the suggested definition (1) of the distribution function $x(\cdot)$ of $\rho(\cdot)$ with center $c = 0$ can be rendered

$$x(t) = (\operatorname{sgn} t)\, \rho(-t^-, t^+], \qquad t \in \mathbb{R}.$$

The RHS makes sense for $q > 1$, cf. 2.3(c), and offers the most convenient definition of the distribution function of a measure $\rho(\cdot)$ over $\mathbb{R}^q$:

2.5. DEF. Let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$. Then the *distribution function* $x(\cdot)$ of $\rho(\cdot)$ (with center 0) is defined by

$$\forall t \in \mathbb{R}^q, \qquad x(t) = (\operatorname{sgn} t)\, \rho(t^-, t^+].$$

It follows at once that the distribution function $x(\cdot)$ of $\rho(\cdot)$ vanishes on all coordinate hyperplanes of $\mathbb{R}^q$ of dimension p, $p = 1, 2, \ldots, q-1$.

2.6. *Remark.* By using a Hamel basis, we can construct a non-measurable function f on $\mathbb{R}$ to $\mathbb{R}$ such that $f(s+t) = f(s) + f(t)$, $t \in \mathbb{R}$, and define $\rho(\cdot)$ on the pre-ring $\mathscr{P}$ over $\mathbb{R}$ given in 2.3(e) by

$$\rho(a, b] = f(b) - f(a), \qquad a \leqslant b. \tag{2}$$

Then $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}, \mathbb{R})$ but its distribution function $x(\cdot) = \pm f(\cdot)$ is non-measurable on $\mathbb{R}$.

It is easily seen that the measure $\rho(\cdot)$ defined in (2) is stationary with the propagator $U(t) = I_{\mathbb{R}}$, cf. [2]. But the nonmeasurability of $x(\cdot)$ destroys the concept of the average vector [2, (2.5), 2.6].[4] In the study of stationary measures, the non-measurability of $x(\cdot)$ is thus a serious pathology, and the following theorem is exceedingly useful:

2.7. MAIN THM. *Let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$ and $x(\cdot)$ be its distribution function. Then following conditions are equivalent:*

(α) *$x(\cdot)$ is continuous on $\mathbb{R}^q$ to $\mathscr{X}$,*

(β) $\forall (a, b] \in \mathscr{P}^q$, $\lim_{s \to a, t \to b} \rho(s, t] = \rho(a, b]$,

(γ) *$\rho(\cdot)$ is ultra-regular on $\mathscr{P}^q$.*

[4] Thus the outline [2] is defective without the initial imposition of a regularity requirement on the measure $\rho(\cdot)$.

3. The Proof of the Main Theorem

We will need several lemmas. We first record one from topology, the proof of which we leave to the reader.

3.1. Lemma. *Let* (i) $\mathscr{X}_1, ..., \mathscr{X}_q$ *be Hausdorff spaces, and* $\mathscr{X} =_d \mathscr{X}_1 \times \cdots \times \mathscr{X}_q$ *be equipped with the weak product topology.*

(a) *If* $K \subseteq \mathscr{X}$ *is compact and* $K \subseteq G_1 \times \cdots \times G_q$, *where* $G_i \subseteq \mathscr{X}_i$ *is open,* $i \in [1, q]$, *then* $\forall i \in [1, q]$, $\exists$ *compact* $K_i \subseteq \mathscr{X}_i$ *such that such that*

$$K \subseteq K_1 \times \cdots \times K_q \subseteq G_1 \times \cdots \times G_q.$$

(b) *If* $G \subseteq \mathscr{X}$ *is open and* $K_1 \times \cdots \times K_q \subseteq G$, *where* $K_i \subseteq \mathscr{X}_i$ *are compact,* $i \in [1, q]$, *then* $\forall i \in [1, q]$, $\exists$ *open* $G_i \subseteq \mathscr{X}_i$ *such that*

$$K_1 \times \cdots \times K_q \subseteq G_1 \times \cdots \times G_q \subseteq G.$$

Our inductive proof, to come, employs the *sectional measures* of $\rho(\cdot)$ on the pre-rings $\mathscr{P}^p$, $p \in [1, q)$, obtained by holding fixed some of the edges of the intervals in $\mathscr{P}^q$. These measures are defined as follows:

3.2. Def. Let (i) $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$, (ii) $J \subset [1, q]$ have p members, (iii) $P_J =_d \{P_j : j \in J\} \subseteq \mathscr{P}$, (iv) ϕ be the one–one, order-preserving function on $J' =_d [1, q] \backslash J$ onto $[1, q-p]$. Then the *P_J-sectional measure* $\rho_{P_J}(\cdot)$ of $\rho(\cdot)$ is defined as follows:

$$\forall\, Q = \mathop{\mathsf{X}}_{i=1}^{q-p} Q_i \in \mathscr{P}^{q-p}, \qquad \rho_{P_J}(Q) = \rho\left(\mathop{\mathsf{X}}_{k=1}^{q} R_k\right),$$

where $R_k = P_k$ for $k \in J$, and $R_k = Q_{\phi(k)}$, for $k \in J'$.

Thus, to evaluate $\rho_{P_J}(\cdot)$ at Q in $\mathscr{P}^{q-p}$, we must evaluate $\rho(\cdot)$ itself at the interval R in $\mathscr{P}^q$, for which the jth edge is P_j for $j \in J$, and the remaining $q-p$ edges are those of Q in their natural order.

The proof that the sectional measures of an $\mathscr{X}$-valued f.a. measure on $\mathscr{P}^q$ are themselves f.a. is slightly technical, since $\mathscr{P}^q$ is not a ring, and therefore it is not enough to check just the 2-additivity of the sectional measure. We shall appeal to the following set-theoretic lemma, the justification of which we leave to the reader:

3.3. Lemma. *Let* (i) Λ *be a non-void set and* $r \in \mathbb{Z}_+$, (ii) $\forall \lambda \in \Lambda$ *and* $\forall i \in [1, r+1]$, B_i^λ, B_i *be sets such that*

$$\bigcup_{\lambda \in \Lambda} \mathop{\mathsf{X}}_{i=1}^{r+1} B_i^\lambda = \mathop{\mathsf{X}}_{i=1}^{r+1} B_i.$$

Then for all sets $A_i,\dots, A_r$,

$$\bigcup_{\lambda\in\Lambda}\left\{\mathop{\times}_{i=1}^{r}(B_i^\lambda\times A_i)\times B_{r+1}^\lambda\right\}=\mathop{\times}_{i=1}^{r}(B_i\times A_i)\times B_{r+1},$$

and the same equality holds when one or both of the end factors B_1^λ, B_1 or B_{r+1}^λ, B_{r+1} are absent.

3.4. Prop. *Let* (i) $\rho(\cdot)\in\mathrm{FA}(\mathscr{P}^q,\mathscr{X})$, (ii) $J\subset[1,q]$ *have p members and $P_J=\{P_j : j\in J\}\subset\mathscr{P}$. Then $\rho_{P_J}(\cdot)\in\mathrm{FA}(\mathscr{P}^{q-p},\mathscr{X})$.*

Proof. Write $\sigma(\cdot)=\rho_{P_J}(\cdot)$ for short, and let $\nu\in\mathbb{Z}_+$, and $Q^1,\dots,Q^\nu\in\mathscr{P}^{q-p}$ be disjoint and such that $Q=_d\bigcup_{n=1}^{\nu}Q^n\in\mathscr{P}^{q-p}$. By Def. 3.2,

$$\sigma(Q)=\rho(R),\qquad R\underset{d}{=}\mathop{\times}_{k=1}^{q}R_k,\qquad R_k\underset{d}{=}\begin{cases}P_k, & k\in J\\ Q_{\phi(k)}, & k\in J'.\end{cases}\tag{1}$$

Now let $J=J_1\cup\cdots\cup J_r$, where the $J_i=[m_i,m_i']$, $m_i\leqslant m'$, are maximal disjoint subintervals included in J from left to right. Let

$$J_1'=[1,m_1),\qquad J_i'=(m_{i-1}',m_i),\qquad 2\leqslant i<r,\qquad J_{r+1}'=(m_r',q].$$

Also, after omitting J_1', J_{r+1}' in case $1=m_1$ or $m_r'=q$, let

$$B_i=\mathop{\times}_{k\in J_i'}Q_{\phi(k)}\qquad\text{and}\qquad A_i=\mathop{\times}_{k\in J_i}P_k.$$

Then, as is easy to check,

$$R=B_1\times A_1\times\cdots\times B_r\times A_r\times B_{r+1}.\tag{2}$$

With an obvious extension of these notations we likewise have in analogy with (1) and (2),

$$\sigma(Q^n)=\rho(R^n),\qquad\text{where}\quad R^n=B_1^n\times\cdots\times B_r^n\times A_r\times B_{r+1}^n.\tag{3}$$

The condition $\bigcup_{n=1}^{\nu}Q^n=Q$ tells us that the premises of the last lemma hold with $\Lambda=[1,\nu]$, and its conclusion tells us that $\bigcup_{n=1}^{\nu}R^n=R$. Since the sets $R^n\in\mathscr{P}^q$ are obviously disjoint, we infer from the finite-additivity of $\rho(\cdot)$ and the relations (1) and (3) that

$$\sigma(Q)=\rho(R)=\sum_{n=1}^{\nu}\rho(R^n)=\sum_{n=1}^{\nu}\sigma(Q^n).\quad\blacksquare$$

The sectional measures $\rho_{p_J}(\cdot)$ of $\rho(\cdot)$ in which J is $[1, p]$ or $[p+1, q]$ are especially useful in arguments based on induction. If $J = [1, p]$ and we let $A =_d \mathsf{X}_{i=1}^{p} P_i \in \mathscr{P}^p$, then

$$\rho_{P_J}(Y) = \rho(A \times Y), \qquad Y \in \mathscr{P}^{q-p}.$$

Likewise, if $J = [p+1, q]$ and we let $B =_d \mathsf{X}_{i=1}^{q-p} P_{i+p} \in \mathscr{P}^{q-p}$, then

$$\rho_{P_J}(X) = \rho(X \times B), \qquad X \in \mathscr{P}^p.$$

It is convenient to have a special notation for these sectional measures:

3.5. DEF. Let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$. Then $\forall p \in [1, q)$ & $\forall A \in \mathscr{P}^p$ & $\forall B \in \mathscr{P}^{q-p}$,

$$\rho_A(B) \underset{d}{=} \rho(A \times B) \underset{d}{=} \rho^B(A).$$

As $\rho_A(\cdot)$, $\rho^B(\cdot)$ are sectional measures, it follows of course from the last proposition that

$$\rho_A(\cdot) \in \mathrm{FA}(\mathscr{P}^{q-p}, \mathscr{X}), \qquad \rho^B(\cdot) \in \mathrm{FA}(\mathscr{P}^p, \mathscr{X}). \tag{3.6}$$

The use of these sectional measures allows us to establish by induction the usual "inclusion–exclusion" formula for the recovery of $\rho(\cdot)$ from its distribution function $x(\cdot)$:

3.7. PROP. *Let* (i) $x(\cdot)$ *be the distribution function of* $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$, and (ii) $(a, b] \in \mathscr{P}^q$. Then

$$\rho(a, b] = x(c_0) - \sum_{i=1}^{q} x(c_i) + \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} x(c_{ij}) - \cdots + (-1)^q x(c_{1\ldots q}),$$

where

$$\begin{aligned} c_0 &= (b_1, \ldots, b_q) = b \\ c_i &= (b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_q) \\ c_{ij} &= (b_1, \ldots, b_{i-1}, a_i, b_{i+1}, \ldots, b_{j-1}, a_j, b_{j+1}, \ldots, b_q) \\ &\vdots \\ c_{1\ldots q} &= (a_1, \ldots, a_q) = a. \end{aligned}$$

Proof. The result is obviously for $q = 1$. Now assume its validity for $q - 1$, let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$, $x(\cdot)$ be the distribution function of $\rho(\cdot)$, and

$(a', b'] \in \mathscr{P}^q$. Let $a' =_d (a, a_q)$, $b' =_d (b, b_q)$, where $a =_d (a_1, ..., a_{q-1})$ and $b =_d (b_1, ..., b_{q-1})$. Then $(a', b'] = (a, b] \times (a_q, b_q]$, and by Def. 3.5

$$\rho(a', b'] = \rho^{(a_q, b_q]}(a, b]. \tag{1}$$

By (3.6), $\rho^{(a_q,b_q]}(\cdot) \in \mathrm{FA}(\mathscr{P}^{q-1}, \mathscr{X})$, and hence by our inductive assumption, RHS (1) can be expanded in an inclusion–exclusion sum, $S_{q-1}[y^q\}$, involving the distribution function $y^q(\cdot)$ of the measure $\rho^{(a_q,b_q]}(\cdot)$ on $\mathscr{P}^{q-1}$; thus

$$\rho(a', b'] = S_{q-1}\{y^q\}. \tag{2}$$

But it is easily verified that $y^q(\cdot)$ on $\mathbb{R}^{q-1}$ is given by

$$\forall t \in \mathbb{R}^{q-1}, \qquad y^q(t) = x(t, b_k) - x(t, a_k). \tag{3}$$

If we make the substitution (3) for y_q in terms of x in the sum $S_{q-1}\{y^q\}$, we will get another expansion for $\rho(a', b']$, involving $x(\cdot)$ alone. We leave it to the reader to check that upon simplification, the latter expansion reduces to the one desired. In this way, we establish the formula for the measure $\rho(\cdot)$ on $\mathscr{P}^q$. By induction, the formula is valid for all $q \in \mathbb{Z}_+$. ▮

The replacement of b by t and a simple transposition immediately yields the following useful corollary:

3.8. Cor. *Let* (i) $x(\cdot)$ *be the distribution function of* $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$, *and* $a \in \mathbb{R}^q$. *Then* $\forall t \in \mathbb{R}^q$ *such that* $a \leqslant t$,

$$\begin{aligned} x(t) = \rho(a, t] &+ \sum_{i=1}^{q} x(t_1, ..., t_{i-1}, a_i, t_{i+1}, ..., t_q) \\ &- \sum_{i=1}^{q-1} \sum_{j=i+1}^{q} x(t_1, ..., t_{i-1}, a_i, t_{i+1}, ..., t_{j-1}, a_j, t_{j+1}, ..., t_q) \\ &+ \\ &\vdots \\ &- (-1)^q x(a). \end{aligned}$$

The next result tells us that the functions $x(..., a_i, ...)$, $x(..., a_i, ..., a_j, ...)$, etc., appearing in the last expansion, are themselves distribution functions of certain sectional measures of $\rho(\cdot)$.

3.9. Prop. *Let* (i) $x(\cdot)$ *be the distribution function of* $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$, (ii) $J \subset [1, q]$ *have* p *members, and* $\phi(\cdot)$ *be the one–one order preserving function on* J *onto* $[1, p]$, (iii) $a \in \mathbb{R}^p$, $\operatorname{sgn} a \neq 0$, *and*

$$P_J = \{(-a^-_{\phi(j)}, a^+_{\phi(j)}] : j \in J\}.$$

Then the distribution function $x_a^J(\cdot)$ *of the sectional measure* $\rho_{P_J}(\cdot)$ *on* $\mathscr{P}^{q-p}$ *is given on* $\mathbb{R}^{q-p}$ *by*

$$x_a^J(\cdot) = (\operatorname{sgn} a)\, x(\ldots, a_1, \ldots, a_2, \ldots, a_p, \ldots),$$

where on the RHS, a_j *occupies the* $\phi^{-1}(j)$th *place.*

Proof. Let $t = (t_1, \ldots, t_{q-p}) \in \mathbb{R}^{q-p}$, $\psi(\cdot)$ be the one–one order-preserving function on $J' =_d [1, q] \backslash J$ onto $[1, q-p]$, and define $s \in \mathbb{R}^q$ by

$$s_k = a_{\phi(k)}, \quad k \in J; \qquad s_k = t_{\psi(k)}, \quad k \in J'. \tag{1}$$

Then from Defs. 3.3 and 3.4, we obtain, after simplification,

$$\begin{aligned} x_a^J(t) &= (\operatorname{sgn} t)\, \rho \left\{ \mathop{\times}_{i=1}^{q} (-s_k^-, s_k^+] \right\} = (\operatorname{sgn} t)\, \rho(-s^-, s^+] \\ &= (\operatorname{sgn} a)(\operatorname{sgn} s)\, \rho(-s^-, s^+] = (\operatorname{sgn} a) \cdot x(s). \end{aligned} \tag{2}$$

Next, we verify that

$$\begin{aligned} s = (t_1, \ldots, t_{m_1 - 1}, a_{m_1}, \ldots, a_{m_1'}, t_{m_1'}, t_{m_1'+1}, \ldots, t_{m_2 - 1}, \ldots, \\ a_{m_r}, \ldots, a_{m_r'}, t_{m_r+1}, \ldots, t_{q-p}). \end{aligned} \tag{3}$$

where $[m_1, m_1'], \ldots, [m_r, m_r']$ are maximal subintervals, from left to right, included in J. In (3), a_j occurs in the $\phi^{-1}(j)$th place, by dint of (1). Replacing s in (2) by the expression in (3), we get the desired expression for $x_a^J(t)$. ∎

The preceding results, though combinatorial in nature, are required to link the ultra-regularity of the measure $\rho(\cdot)$ with the continuity of its distribution function $x(\cdot)$. We now turn to this linkage. The following simple lemma is needed:

3.10. Lemma. *Let* $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$ *be such that*

$$\forall (a, b] \in \mathscr{P}^q, \qquad \lim_{\substack{s \to a \\ t \to b}} \rho(s, t] = \rho(a, b].$$

Then every sectional measure $\sigma(\cdot)$ *of* $\rho(\cdot)$ *on* $\mathscr{P}^r$, $1 \leqslant r < q$, *has the same property, i.e.,*

$$\forall (a, b] \in \mathscr{P}^r, \qquad \lim_{\substack{s \to a \\ t \to b}} \sigma(s, t] = \sigma(a, b].$$

Proof. Let $\sigma(\cdot) = \rho_{P_J}(\cdot)$, where, cf. Def. 3.4, $J \subset [1, q]$ has p members and $P_J = \{(c_j, d_j] : j \in J\} \subset \mathscr{P}$, and let ϕ be the one–one order-preserving function on $J' =_d [1, q]\backslash J$ onto $[1, q-r]$. For *any* $(s, t] \in \mathscr{P}^{q-p}$, define $(s', t'] \in \mathscr{P}^q$ by

$$s'_k = c_k, \qquad t'_k = d_k, \qquad \forall k \in J$$
$$s'_k = s_k, \qquad t'_k = t_k, \qquad \forall k \in J'.$$

Then by Def. 3.4,

$$\forall (s, t] \in \mathscr{P}^{q-p}, \qquad \sigma(s, t] = \rho(s', t']. \tag{1}$$

It is also obvious that $\forall (a, b] \in \mathscr{P}^{q-p}$,

$$s \to a \text{ and } t \to b \text{ in } \mathbb{R}^{q-p} \qquad \text{iff} \quad s' \to a' \text{ and } t' \to b' \text{ in } \mathbb{R}^q,$$

and so by (1) and the hypothesis on $\rho(\cdot)$,

$$\lim_{\substack{s \to a \\ t \to b}} \sigma(s, t] = \lim_{s' \to a'} \rho(s', t'] = \rho(a', b'] = \sigma(a, b]. \quad \blacksquare$$

We are now ready to prove the Main Thm. 2.7 itself.

Proof of Main Thm. 2.7. We shall show that $(\alpha) \Leftrightarrow (\beta)$ and $(\beta) \Leftrightarrow (\gamma)$.

To prove that $(\alpha) \Rightarrow (\beta)$, let (α) hold. Then $x(\cdot)$ as well as all its restrictions are continuous. More precisely, we have for $1 \leqslant i < j < k < \cdots$ and $c_i, c_j, c_k, \ldots \in \mathbb{R}$,

$$\left.\begin{aligned} x(\ldots) &\in C(\mathbb{R}^q, \mathscr{X}) \\ x(\ldots, c_i, \ldots) &\in C(\mathbb{R}^{q-1}, \mathscr{X}) \\ x(\ldots, c_i, \ldots, c_j, \ldots) &\in C(\mathbb{R}^{q-2}, \mathscr{X}) \end{aligned}\right\} \tag{1}$$

and so on.

Now let $(s, t]$ and $(a, b] \in \mathscr{P}^q$, and consider the inclusion–exclusion formula for $\rho(s, t]$ given in 3.7. When $t \to b$, the terms in that formula will, by (1), approach

$$x(b_1, \ldots, b_q)$$
$$x(s_1, \ldots, s_{i-1}, b_i, s_{i+1}, \ldots, s_q)$$
$$x(s_1, \ldots, s_{i-1}, b_i, s_{i+1}, \ldots, s_{j-1}, b_j, s_{j+1}, \ldots, s_q)$$

and so on. These terms in turn, when $s \to a$, will approach

$$x(b_1, \ldots, b_q)$$
$$x(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_q)$$
$$x(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_{j-1}, b_j, a_{j+1}, \ldots, a_q)$$

etc., which are the terms in the inclusion–exclusion formula for $\rho(a, b]$. We thus see that $\rho(s, t] \to \rho(a, b]$, as $s \to a$ and $t \to b$. Thus (β) is established.

To prove that $(\beta) \Rightarrow (\alpha)$, we note that for $q = 1$, this implication follows at once from the equation

$$\rho(a, t] = x(t) - x(a), \qquad \forall (a, t] \in \mathscr{P}.$$

Now assume that the implication $(\beta) \Rightarrow (\alpha)$ holds for $p = 1, 2, \ldots, q-1$, and let $x(\cdot)$ be the distribution function of a measure $\rho \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$ satisfying (β), and let $b \in \mathbb{R}^q$. Pick an a in $\mathbb{R}^q$ such that $a < b$. By Lemma 3.10, all the sectional measures of $\rho(\cdot)$ satisfy the condition (β) on their domains $\mathscr{P}^p$, $1 \leqslant p < q-1$, and therefore by our inductive assumption, their distribution functions are continuous on the spaces $\mathbb{R}^p$. Now by Prop. 3.9, the restrictions of $x(\cdot)$ appearing on the RHS of the formula for $x(t)$ in Cor. 3.8 are indeed distribution functions of certain sectional measures of $\rho(\cdot)$. These restrictions are therefore continuous on $\mathbb{R}^{q-1}$, $\mathbb{R}^{q-2}, \ldots, \mathbb{R}^1$. Since by ($\beta$), $\lim_{t \to b} \rho(a, t] = \rho(a, b]$, it follows on letting $t \to b$ in the formula in 3.8 that $x(t) \to x(b)$, i.e., $x(\cdot)$ satisfies (α). The implication $(\beta) \Rightarrow (\alpha)$ thus prevails for $p = q$. By induction it prevails for all $q \in \mathbb{Z}_+$.

To show that $(\beta) \Rightarrow (\gamma)$, let ($\beta$) hold, $(a, b] \in \mathscr{P}^q$ and $\varepsilon \in \mathbb{R}_+$. Then $\exists \delta_0 \in \mathbb{R}_+$ such that

$$|s - a| < \delta_0 \text{ and } |t - b| < \delta_0 \Rightarrow |\rho(s, t] - \rho(a, b]| < \varepsilon. \tag{2}$$

Now let $\delta = (\delta_1, \ldots, \delta_q)$, where each $\delta_i = \delta_0 / \sqrt{q}$ and let

$$K =_d [a + \delta, b - \delta], \qquad U = (a - \delta, b + \delta).$$

Then K and U are compact and open subsets of $\mathbb{R}^q$ satisfying

$$K \subseteq (a, b) \subseteq [a, b] \subseteq U,$$

and it is easy to see that

$$\begin{aligned}(s, t] \in \mathscr{P}^q \,\&\, K \subset (s, t] \subset U &\Rightarrow |s - a| < \delta_0 \,\&\, |t - b| < \delta_0 \\ &\Rightarrow |\rho(s, t] - \rho(a, b]| < \varepsilon, \qquad \text{by (2).}\end{aligned}$$

Thus $\rho(\cdot)$ is regular, i.e., we have (γ).

Finally, to show that $(\gamma) \Rightarrow (\beta)$, let ($\gamma$) hold, let $P =_d (a, b] \in \mathscr{P}^q$ and $\varepsilon > 0$. Then there exists a compact subset K and an open subset U of $\mathbb{R}^q$ such that

$$\left.\begin{array}{l} K \subseteq (a, b) \subseteq [a, b] \subseteq U \\ \text{and} \\ Q \in \mathscr{P}^q \,\&\, K \subseteq Q \subseteq P \Rightarrow |\rho(Q) - \rho(P)| < \varepsilon. \end{array}\right\} \tag{3}$$

We maintain that $\exists\delta = (\delta_1, \ldots, \delta_q) \in \mathbb{R}_+^q$ such that

$$K \subset [a+\delta, b-\delta] \subset (a, b] \subset (a-\delta, b+\delta) \subseteq U. \tag{I}$$

The proof is topological. By Lemma 3.1, we obtain compact sets $K_1, \ldots, K_q \subset \mathbb{R}$ and open sets $U_1, \ldots, U_q \subset \mathbb{R}$ such that

$$K \subseteq \mathop{\times}_{i=1}^{q} K_i \subset (a, b) \subset [a, b] \subset \mathop{\times}_{i=1}^{q} U_i \subseteq U. \tag{4}$$

This shows that

$$\forall i \in [1, q], \qquad K_i \subseteq (a_i, b_i) \subset [a_i, b_i] \subset U_i.$$

Obviously for each $i \in [1, q]$, we can find $\delta_i \in \mathbb{R}_+$ such that

$$\begin{aligned} K_i \subset (a_i + \delta_i, b_i - \delta_i) \subset (a_i, b_i) \subset [a_i, b_i] \\ \subset (a_i - \delta_i, b_i + \delta_i) \subseteq U_i. \end{aligned}$$

Taking the Cartesian product and appealing to (4) we get (I).

Now let $\delta_0 = \min\{\delta_1, \ldots, \delta_q\}$. Then it is readily seen that

$$\begin{aligned} &|s-a| < \delta_0 \text{ and } |t-b| < \delta_0 \\ &\quad \Rightarrow a-\delta < s < a+\delta < b-\delta < t < b+\delta \\ &\quad \Rightarrow [a+\delta, b-\delta] \subset (s, t] \subset (a-\delta, b+\delta) \\ &\quad \Rightarrow K \subset (s, t] \subset U, && \text{by } (I) \\ &\quad \Rightarrow |\rho(s, t] - \rho(a, b]| < \varepsilon, && \text{by (3).} \end{aligned}$$

Thus

$$\lim_{\substack{s \to a \\ t \to b}} \rho(s, t] = \rho(a, b],$$

i.e., we have (β). ∎

An immediate corollary of the main theorem is the result that the sectional measures of a regular measure are regular:

3.11. Cor. *Let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}^q, \mathscr{X})$ and $\sigma(\cdot)$ be a sectional measure of $\rho(\cdot)$ on $\mathscr{P}^r$, $1 \leqslant r < q$. Then*

$$\rho(\cdot) \textit{ is regular on } \mathscr{P}^q \Rightarrow \sigma(\cdot) \textit{ is regular on } \mathscr{P}^r.$$

Proof. If $\rho(\cdot)$ is regular, then by the main theorem, it satisfies the condition (β). Hence by Lemma 3.10 $\sigma(\cdot)$ satisfies (β), and hence by the main theorem $\sigma(\cdot)$ is regular. ∎

4. The Right Continuity of the Distribution Function

It is worth noting that the following exact analogue of the Main Thm. 2.7 prevails:

4.1. Thm. *Let $\rho(\cdot) \in \mathrm{FA}(\mathscr{P}, \mathscr{X})$ and $x(\cdot)$ be its distribution function. Then the following conditions are equivalent:*

(α) *$x(\cdot)$ is right-continuous*[5] *on $\mathscr{P}^q$ to $\mathscr{X}$,*

(β) $\forall (a, b] \in \mathscr{P}^q, \lim_{\substack{s \to a+ \\ t \to b+}} \rho(s, t) = \rho(a, b]$,

(γ) *$\rho(\cdot)$ is regular on $\mathscr{P}^q$.*

The proof is even more technical than that of Thm. 2.7, and will be given elsewhere.

References

1. N. Dinculeanu, "Vector Measures," Pergamon, Elmsford, N. Y., 1967.
2. P. Masani, An outline of the theory of stationary measures over $\mathbb{R}^q$, *in* "Measure Theory" (D. Kolzöw, Ed.), pp. 295–309, Lecture Notes in Mathematics No. 794, Springer-Verlag, New York/Berlin, 1980.
3. J. von Neumann, "Functional Operators, I," Princeton Univ. Press, Princeton, N. J., 1950.

[5] That is, $\lim_{t \to a+} x(t) = x(a)$, $\forall a \in \mathbb{R}^q$. We use the same definition of right-limit, as $t \to a+$, of functions on $\mathbb{R}^q$, as von Neumann [3, p. 166, No. 10.5.4].